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Asymptotic Behavior of a Convergent Spherical Rarefaction Wave

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An ideal gas with isentropic exponent $\mathscr{H}$ surrounds an impenetrable sphere inside of which there is a vacuum at time $t \leqslant t_0$. The distribution of gas dynamic values at t_0 has finite gradients. At t_0 the sphere disappears, and a strong rarefaction wave arises, bounded by a free surface with zero pressure and by a characteristic whose dynamic functions coincide with those in the undisturbed region. The free boundary attains a velocity which remains constant until any characteristic or the shock wave reaches it.

When the isentropic exponent $\mathscr{H} < \frac{5}{3}$, the result is that the velocity of the free boundary is constant until the moment of focussing, i.e., until the instant when the free boundary arrives at the center and a reflected shock wave goes out from the center. When $\mathscr{H} \geqslant \frac{5}{3}$ the free boundary begins to accelerate at a certain moment t_1. When $\mathscr{H} \geqslant 3$, t_1 coincides with t_0.

INTRODUCTION

The work is divided into 3 parts:

(1) Discussion of the asymptotic behavior of the rarefaction wave at the beginning of the process.

(2) Investigation of the asymptotic behavior in the neighborhood of the free boundary.

(3) Discussion of the asymptotic behavior at the moment of focussing when $\mathscr{H} \leqslant 5/3$, and at the beginning of the acceleration when $\mathscr{H} > 5/3$.

As the entropy in the rarefaction wave is constant, one can consider the gas isentropic until the shock wave appears. After similarity and shift transformations the process can be considered to start at $t_0 = -1$, with an initial velocity of the free boundary of $u_0 = -1$, and the equation of state of

$$P = v^{-\mathscr{H}}, \tag{1}$$

where v means the specific volume. The equations of gas dynamics are in Lagrangian form,

$$\frac{\partial r}{\partial t} = u, \qquad v = \frac{\partial r^3}{\partial m}, \qquad \frac{\partial u}{\partial t} = -3r^2 \frac{\partial P}{\partial m}. \tag{2}$$

1. The centered rarefaction wave is plane in its leading term close to t_0. Therefore it is selfsimilar and the asymptotic behavior is determined by the series expansion in powers of $t + 1$, when $t + 1$ is sufficiently small. The coefficients of this expansion are functions of the selfsimilar variable $\xi = m/t + 1$, reckoned from the free boundary, where m is mass. Thus

$$f = f_0(\xi) + f_1(\xi)(t+1) + f_2(\xi)(t+1)^2 + \dots . \tag{3}$$

In particular, when $\mathscr{H} \neq 5/3$, $\mathscr{H} \neq 3$

$$\begin{aligned} r' &= 1 + (A_{11}\xi^{1/h} - 1)(1+t) + (A_{21}\xi^{1/h} + A_{22}\xi^{2/h} + C\xi^{1/2})(1+t)^2 + \cdots, \\ u &= -1 + B_{11}\xi^{1/h} + (B_{21}\xi^{1/h} + B_{22}\xi^{2/h} + \tfrac{1}{2}C\xi^{1/2})(1+t) + \cdots, \\ \xi v &= C_{11}\xi^{1/h} + (C_{21}\xi^{1/h} + C_{22}\xi^{2/h} + \tfrac{3}{2}C\xi^{1/2})(1+t) + \cdots, \end{aligned} \tag{4}$$

where $h = (\mathscr{H} + 1)/(\mathscr{H} - 1)$, and A_{ij}, B_{ij}, C_{ij} are coefficients which depend on $\mathscr{H}$. The value of C is determined by the gradients in the undisturbed gas. When $h = 4$, ($\mathscr{H} = 5/3$), and $h = 2$, ($\mathscr{H} = 3$) these formulae degenerate; two neighboring terms have the same exponent and the coefficient of one of them becomes infinite. For these exceptional exponents the asymptotic behavior is changed and logarithmic terms appear. It is important to note that the first n terms of the asymptotic behavior have been obtained with the assumption that

$$(t+1) f_n(\xi)/f_{n-1}(\xi) \to 0 \tag{5}$$

when $(t + 1) \to 0$.

Since in the centered rarefaction wave the variable ξ varies from 0 (corresponding to the free boundary) up to a finite value (determined by the initial data), the condition represented by Eq. (5) is fulfilled uniformly in the whole region of the intersection of the neighborhood of the point ($m = 0$, $t = -1$) with the rarefaction wave, provided $1 < \mathscr{H} < 3$. When $\mathscr{H} \geqslant 3$, i.e., $h < 2$ and $\xi \to 0$,

$$f_2(\xi)/f_1(\xi) \sim \xi^{(h-2)/2h} \to \infty,$$

the condition (5) is not fulfilled at $\xi = 0$, and the determination of the asymptotic behavior in the neighborhood of the point ($m = 0$, $t = -1$) requires additional investigation.

2. Asymptotic behavior in the neighborhood of the free boundary, $1 < \mathscr{H} < 3$.

Taking into account the asymptotic equations (4), which are valid in the whole rarefaction wave when $t + 1 \to 0$, we shall look for an asymptotic expansion in the neighborhood of the free boundary, i.e., when ξ is small and t is finite, of the form:

$$\begin{aligned} r &= -t + \phi_1(t)\,\xi^{1/h} + \psi_1(t)\,\xi^{\beta} + \cdots, \\ u &= -1 + \phi_2(t)\,\xi^{1/h} + \psi_2(t)\,\xi^{\beta} + \cdots, \\ \xi v &= \phi_3(t)\,\xi^{1/h} + \psi_3(t)\,\xi^{\beta} + \cdots, \end{aligned} \tag{6}$$

where $\beta = 2/h$ when $1 < \mathscr{H} < 5/3$ and $\beta = 1/2$ when $\mathscr{H} > 5/3$. Substituting Eq. (6) into Eq. (2), we shall obtain a system of equations for ϕ and ψ. In each case there are two differential equations and an algebraic relation between these functions. The initial condition can be obtained from the requirement that the asymptotic solutions of Eq. (6) and (4) must match when $1 + t \to 0$ and $\xi \to 0$. The equations and initial asymptotic expansion which determine the functions $\phi_i(t)$ are:

$$h(1+t)\,\phi_1' - \phi_1 = h(1+t)\,\phi_2, \quad h(1+t)\,\phi_2' - \phi_2 = -3(h+1)\,t^2\phi_3^{-(\mathscr{H}+1)}$$
$$h(1+t)\,\phi_3 = 3t^2\phi_1, \qquad t \to -1 : \phi_1 \sim (1+t), \quad \phi_2 \sim \text{const}, \quad \phi_3 \sim \text{const}. \tag{7}$$

This system is not linear. After change of variables we arrive at the following conclusion. The function ϕ_1 (and hence ϕ_3) is bounded and different from zero when $-1 < t < 0$ if $1 < \mathscr{H} \leqslant 5/3$. If $\mathscr{H} > 5/3$, then there is a point t_1, $-1 < t_1 < 0$ such that $\phi_1(t_1) = 0$. The function $\phi_2(t)$ is bounded within the same time intervals. But it need not be bounded when $\mathscr{H} \geqslant 3/2$.

$$\begin{aligned} t \to 0; \quad & \mathscr{H} < 3/2 : \phi_2(t) \to \text{const}, \qquad \mathscr{H} = 3/2 : \phi_2(t) \sim \ln(-t), \\ & 3/2 < \mathscr{H} < 5/3 : \phi_2(t) \sim (-t)^{3-2\mathscr{H}} \\ t \to t_1; \quad & 5/3 < \mathscr{H} < 3 : \phi_2(t) \sim (t_1 - t)^{-1/h}. \end{aligned} \tag{8}$$

The functions $\psi_i(t)$ are determined by a linear system of equations which depend on $\phi_i(t)$. Analysis of this equation leads to the result that the functions $\phi_i(t)$ are bounded when $-1 < t < 0, \mathscr{H} \leqslant 5/3$ and when $-1 < t < t_1$, $\mathscr{H} > 5/3$. They have the asymptotic behavior

$$\begin{aligned} & 1 < \mathscr{H} < 5/3, \quad t \to 0; \quad \psi_1(t) \sim t^{3-2\mathscr{H}}, \qquad \psi_2(t) \sim t^{2-2\mathscr{H}}, \qquad \psi_3(t) \sim t \\ & 5/3 < \mathscr{H} < 3, \quad t \to t_1; \quad \psi_1(t) \sim (t_1 - t)^{-1/2}, \qquad \psi_2(t) \sim (t_1 - t)^{-3/2}, \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \psi,(t) \sim (t, - t)^{-1/2}. \end{aligned} \tag{9}$$

The first terms of Eq. (6) represent the asymptotic expansion of the corresponding functions only in the domain where the ratios of $\psi_1\xi^{1/h}/t$, $\phi_2\xi^{1/h}$, $\psi_3\xi^{\beta-1/h}/\phi_3$ are small enough. Such domains coincide with the domain where values of the quantities $\xi^{1/h}/t$ are small $(-1 < t < 0)$ when $\mathscr{H} < 5/3$, and where

$$\xi^{h-2}(t_1 - t)^{2-3h}, (-1 < t < t_1)$$

are small when $\mathscr{H} > 5/3$. Further analysis shows that in these domains one can consider Eq. (6) as the asymptotic representations of the corresponding functions. Since in the domains just referred to

$$|\,\phi_2\xi^{1/h} + \psi_2\xi^{\beta}\,| \to 0$$

when

$$\xi \to 0,$$

the velocity of the free boundary during the corresponding time intervals $-1 < t < 0$ ($\mathscr{H} \leqslant 5/3$) and $-1 < t < t_1$($\mathscr{H} > 5/3$) is constant: $u = -1$. This result may be illustrated by the fact that no characteristic during these intervals reaches the free boundary.

It will be useful now to summarize the asymptotic expansion obtained in the neighborhood of the free boundary when $t \to 0$, $\mathscr{H} < 5/3$ and when $t \to t_1$, $\mathscr{H} > 5/3$.

(a) $\mathscr{H} < 5/3$, $t \to 0$;

$$r = -t + c_1 m^{1/2} + \cdots, \qquad v \approx 3c_1 t^2 m^{h-1/h},$$

$$u \approx -1 + \begin{cases} m^{1/h} & \text{when } \mathscr{H} < 3/2 \\ m^{1/h} \ln(-t) & \text{when } \mathscr{H} = 3/2 \\ m^{1/h}(-t)^{3-2\mathscr{H}} & \text{when } \mathscr{H} > 3/2 \end{cases}$$

(b) $\mathscr{H} > 5/3$, $t \to t_1$;

$$\begin{aligned} r &\approx -t_1 + Ah(t_1 - t)^{(h-1)/h} m^{1/h} + LC(t_1 - t)^{-1/2} m^{1/2} \\ u &\approx -1 + A(h-1)(t_1 - t)^{-1/h} m^{1/h} + 1/2LC(t_1 - t)^{-3/2} m^{1/2} \\ v &= 3At_1^2(t_1 - t)\, m^{1/h-1} + 3/2t_1^2 LC(t_1 - t)^{-1/2} m^{-1/2}, \qquad L > 0. \end{aligned} \tag{10}$$

C is a constant, depending on the gradients at $t = -1$.

In the rt plane the asymptotic expansions are valid only in the domain limited by the curve tangent to the free boundary at the center ($m = 0, t = 0$), when $\mathscr{H} < 5/3$, or at the point ($m = 0$, $t = t_1$), when $\mathscr{H} > 5/3$, but they influence the asymptotic behavior in the whole neighborhood of these points, and determine the choice of a typical selfsimilar variable for a given neighborhood.

3.1. *Asymptotic behavior in the neighborhood of the center when t is sufficiently close to the moment of focussing and* $\mathscr{H} < 5/3$.

It is obvious that the principal terms of the asymptotic expansion which is to be found must coincide with that obtained earlier near the free boundary ($m = 0$):

$$r \approx -t,\ u \approx -1,\ v \approx 3c_1 t^2 m^{-2/\mathscr{H}+1}/h \text{ when } t \to 0$$

and

$$\zeta = m^{1/h}/t,\ t < 0,$$

is small enough. So it is natural to search for asymptotic representations in the whole neighborhood of the following form:

$$r = m^{1/h} R(\zeta), \qquad u = u(\zeta), \qquad v = m^{1/h-1/3} V(\zeta), \qquad \zeta = m^{1/h}/t. \tag{11}$$

Since the gas is isentropic

$$P = m^{(h-3)\mathscr{H}/h}[v(\zeta)]^{-\mathscr{H}}. \tag{12}$$

In this representation, $m \to 0$, ζ is arbitrary and the functions themselves are:

$$R \approx -1/\zeta + c_1, \qquad V \approx 3c_1/hR^2, \qquad u \approx -1. \tag{13}$$

It is obvious that this solution is valid only in a part of the neighborhood, because r becomes negative when $\zeta \to 0$ $(t > 0)$.

The reflected shock wave in our approximation corresponds to a line $\zeta = \text{const}$. Because of the conditions at the shock front, the functions r, u, v preserve their representation (11) behind the front and Euler's equation yields the following representation for P:

$$P = m^{(h-3)/3h}\,\mathscr{P}(\zeta). \tag{14}$$

Behind the front the flow is not isentropic so we are not able to evaluate P from Eq. (12). The unknown functions are determined by a system of ordinary differential equations; the boundary conditions are determined by the conditions at the shock wave and by the fact that the velocity at the center is zero. The solution in the neighborhood of the center gives the distribution of gas dynamic quantities at the moment of focussing ($t = 0$) when $m \to 0$:

$$r \sim m^{1/h}, \qquad u \sim -1, \qquad v \sim m^{3-h/h}, \qquad P \sim m^{(h-3)\mathscr{H}/h}, \tag{15}$$

and the values of the gas dynamic functions at the center when $t > 0$:

$$P \sim t^{h-3}, \qquad Pv^{\mathscr{H}} \sim m^{2(\mathscr{H}-2)/\mathscr{H}h}, \tag{16}$$

i.e., the pressure is finite, the entropy and the specific volume are infinite.

3.2. *Asymptotic representation in the neighborhood of the point* ($m = 0, t = t_1$), *i.e., at the beginning of acceleration of the free boundary* ($5/3 < \mathscr{H} < 3$).

The asymptotic expansion written above for these values of $\mathscr{H}$ in the neighborhood of the free boundary at $-1 < t < t_1$ are valid if $m^{h-2}(t_1 - t)^{2-3h}$, $(t < t_1)$ is small enough or (which is equivalent) the modulus of $z = m^{h-2/2-3h}/(t_1 - t)$ is very large. For this reason the variable z is characteristic of the neighborhood of the point ($m = 0, t = t_1$). We shall look for asymptotic representation in this neighborhood of the following form:

$$\begin{aligned} r &\approx -t + A\,|(t - t_1)^{h/h-2}\, z^{2-3h/h(h-2)}\,|\,f_1(z) \\ u &\approx -1 - (h-1)/h\,A\,|(t - t_1)^{2/h-2}\, z^{(2-3h)/h(h-2)}\,|\,f_2(z) \\ v &\approx 3t_1^2A/h\,|(t - t_1)^{2(1-h)/(h-2)}\, z^{(1-h)(2-3h)/h(h-2)}\,|\,f_3(z), \end{aligned} \tag{17}$$

where z varies from $-\infty$ to ∞ and the values $z = \pm\infty$ correspond to the free boundary (the minus sign applies when $t < t_1$, and the plus sign when $t > t_1$).

The Eqs. (17) determining f_1, f_2, f_3 can be reduced to one differential equation, a quadrature, and an algebraic relation. The unique solution of this system of equations is determined by the asymptotic behavior when $z \to -\infty$ and by the requirement that the pressure $P \to 0$ when $z \to \infty$. The sign of the constant C, which depends on the gradients in the undisturbed gas, is now very important. If $C > 0$, the required solution exists only if the integral curve has a discontinuity in the first derivative at the singular point of the system, corresponding to a characteristic. This means that the gas dynamic functions have weak discontinuities along the characteristic which originates at the point ($q = 0$, $t = t_1$). From $t = t_1$ onwards, the characteristics *catch up* with the free boundary and reflect from it. The asymptotic representations of the gas dynamic functions in the neighborhood of the free boundary when $t > t_1$, are:

$$r \approx -t + R_0(t - t_1)^{h/h-2}, \qquad u \approx -1 + u_0(t - t_1)^{2/h-2},$$
$$v \approx v_0 m^{-1/\mathscr{H}}(t - t_1)^{(h-4)/\mathscr{H}(h-2)}. \tag{18}$$

It follows from these formulae that from $t = t_1$ onwards the free boundary accelerates and one can expect that near the moment of focussing the flow will correspond to the selfsimilar regime of the collapsing cavity in the neighborhood of the center [1].

It follows from the analysis of the system for $f_i(z)$, that if $C < 0$ the solution becomes nonunique. The characteristics of the same family in the plane mt intersect each other when $t < t_1$. There exists an envelope of this family of characteristics, which coincides with the line $z = \text{const}$, and therefore a shock wave appears. The shock goes through the rarefaction wave and reaches the free boundary. The corresponding regime has been studied in [2] and [3]. It is obvious that further study of the asymptotic behavior in the neighborhood of the free boundary when $t > t_1$ is impossible, if we use the analytical means applied here, because this asymptotic behavior is determined by those characteristics which originate at a finite interval in the region $t = -1$.

3.3. *Asymptotic representations of gas dynamic functions in the neighborhood of the free boundary at the beginning of the process when $\mathscr{H} > 3$.*

It has been shown previously that, for the values of $\mathscr{H}$ stated, the asymptotic expansions in the neighborhood of the point ($m = 0$, $t = -1$) are valid only when $\xi^{h-2/2h}(1 + t)$ is small enough, i.e., when $\lambda = m(1 + t)^{(h+2)/(h-2)}$ is large ($h < 2$ here). Therefore, it is natural to seek asymptotic solutions in the remaining part of the neighborhood of the point ($m = 0$, $t = -1$) in the form of functions depending on λ. When $\lambda \to \infty$ these asymptotic representations must coincide with those obtained earlier in Eq. (4). Therefore the gas dynamic functions in this part of the neighborhood have the form:

$$r \approx -t + A_1(1+t)\,\xi^{1/h} f_1(\lambda), \qquad u \approx -1 + B_1 \xi^{1/h} f_2(\lambda),$$
$$\xi v \approx C_1 \xi^{1/h} f_3(\lambda). \tag{19}$$

The variable λ extends from 0 to ∞. The value $\lambda = 0$ corresponds to the free boundary. Here, as in the case $5/3 < \mathscr{H} < 3$, the sign of the constant c, which is determined by the gradients of the initial distribution in the gas, is very important. If $C > 0$ then the distribution of gas dynamic functions near the free boundary (when $1 + t$ is small) has the form:

$$r \approx -t + Km^{2/h+2}(1+t), \qquad u \approx -1 + Km^{2/h+2},$$
$$v \approx 2/4 + 2\,Km^{-h/h+2}(1+t). \tag{20}$$

If $C < 0$ the gas dynamic functions have a weak discontinuity along the characteristic which coincides with the line $\lambda = \text{const}$, because of the requirement that the pressure vanishes at the free boundary. Asymptotic expansions in the neighborhood of the free boundary have the form:

$$r \approx -t + \mathscr{D}_1(1+t)^{(h-4)/(h-2)}, \qquad u \approx -1 + \mathscr{D}_2(1+t)^{2/2-h},$$
$$v \approx \mathscr{D}_3 m^{-1/\mathscr{H}}(1+t)^{h/(h-2)\mathscr{H}}. \tag{21}$$

In this case the free boundary starts accelerating just at the beginning of the process.

It is important to note that the characteristics *catch up* with the free boundary from $t = -1$ onwards independent of the sign of C. Therefore, further investigation of the asymptotic behavior in the neighborhood of the free boundary, when $t + 1$ is finite, seems to be impossible by the methods applied here. The asymptotic solutions obtained here by analytical means were confirmed by the numerical solution of the corresponding problems by Godunov. The difference scheme was specially accomodated to allow for the calculation of rarefaction waves.

References

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